Failure Time Models

Henrique Laureano
http://leg.ufpr.br/~henrique

January 28, 2020

Failure Time Models,
2nd chapter of *The Statistical Analysis of Failure Time Data*
Kalbfleisch and Prentice, 2002

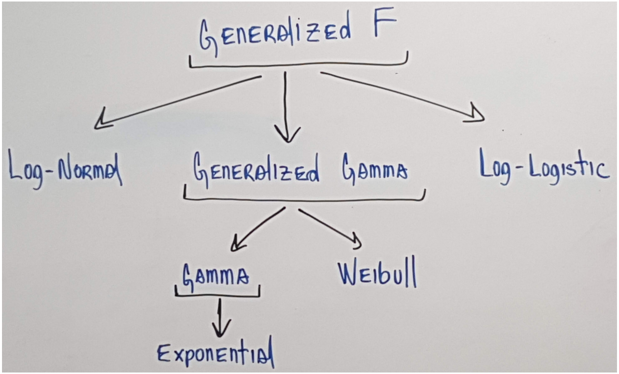
Outline

- Some continuous parametric failure time models
- Regression models
  - Exponential and Weibull
  - Relative Risk or Cox Model
  - Accelerated failure time model
- Discrete failure time models

Some continuous parametric failure time models

Random variable: failure time, $T > 0$.
Type: continuous.

Common failure time distributions for homogeneous populations:



Now, let's better understand how this works.

Generalized F

Advantage: it can adapt to a wide variety of distributional shapes.

Context

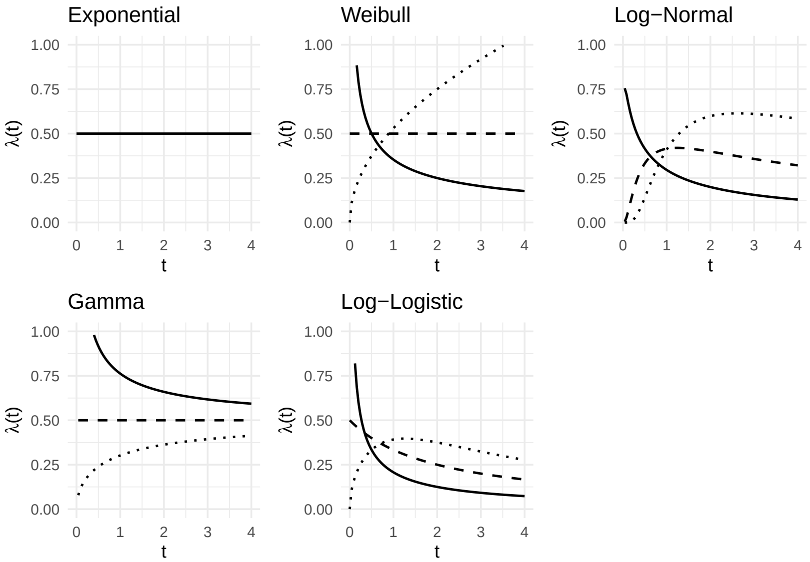
A location and scale model for $Y = \log T$ in which the error distribution is assumed to be that of the logarithm of an F variate on $2m_1$ and $2m_2$ degrees of freedom.

That is, $Y = \mu + \sigma W$, where

$$f_W(w) = \frac{(m_1/m_2)^{m_1} e^{wm_1} (1 + m_1 e^w / m_2)^{-(m_1+m_2)}}{B(m_1, m_2)}.$$

The resulting model for T is the generalized F distribution.

Shapes of the hazard functions



A door to another world

To be able to see these Generalized F special cases,
the transformation $Y = \mu + \sigma W$ is necessary.

However, this open a door for another world: Extreme Value Theory.

In the Generalized Gamma case and special cases,
W is an extreme value (minimum) distribution.

Extreme Value Theory
- ↳ Generalized extreme value (GEV) distribution
  - ↳ Type I extreme value distribution: Gumbel family
  - ↳ Type II extreme value distribution: Fréchet family
  - ↳ Type III extreme value distribution: Weibull family

Regression models
↳ Exponential and Weibull

Goal: obtain a regression model by allowing the failure rate to be a function of the derived covariates Z.

The hazard at time t for an individual can be written as

$$\lambda(t; x) = \text{hazard} \times c(Z^\top \beta),$$

three forms have been used for c:

- $c(s) = 1 + s$, corresponding to the failure rate;
- $c(s) = (1 + s)^{-1}$, corresponding to the mean survival time;
- $c(s) = \exp(s)$.

Exponential regression model

$$\lambda(t;x) = \lambda \exp(Z^\top \beta)$$
$$Y = -\log \lambda - Z^\top \beta + W$$
$$W \sim \text{Extreme Value dist.}$$

Weibull regression model

$$\lambda(t;x) = \gamma(\lambda t)^{\gamma - 1} \exp(Z^\top \beta)$$
$$Y = -\log \lambda - Z^\top \sigma \beta + \gamma^{-1} W$$
$$W \sim \text{Extreme Value dist.}$$

Accelerated failure time models
- ↳ general class of log-linear models
  - ↳ covariates act additively on Y, or multiplication on T
    - ↳ log survival time, $Y = \log T$

More general model: Relative Risk or Cox Model.

Relative risk model

Cox, 1972

$$\lambda(t;x) = \lambda_0(t)\exp(Z^\top \beta),$$

where $\lambda_0(\cdot)$ is an arbitrary unspecified baseline hazard function for continuous T.

The conditional survivor function for T given Z is

$$F(t;x) = F_0^{\exp(Z^\top \beta)}(t), \quad \text{where} \quad F_0(t) = \exp\left[-\int_0^t \lambda_0(u)\mathrm{d}u\right].$$

Thus the survivor function of t for a covariate value, x, is obtained by raising the baseline survivor function $F_0(t)$ to a power.

Nice generalizations,

- stratified Cox model;
- time-dependent Cox model: *relative* risk model.

Accelerated failure time model

Suppose $Y = \log T$ and consider the linear model

$$Y = Z^\top \beta + W.$$

Exponentiation gives $T = \exp(Z^\top \beta)\, S$, where $S = \exp(W) > 0$ has hazard function $\lambda_0(s)$, say, that is independent of β.

The hazard function for T can be written as

$$\lambda(t; x) = \exp(-Z^\top \beta)\lambda_0[t \exp(-Z^\top \beta)].$$

The effect of the covariate is multiplicative on t rather than on the hazard function.

i.e.,

The role of Z is to accelerate (or decelerate) the time to failure.

Comparison of regression models

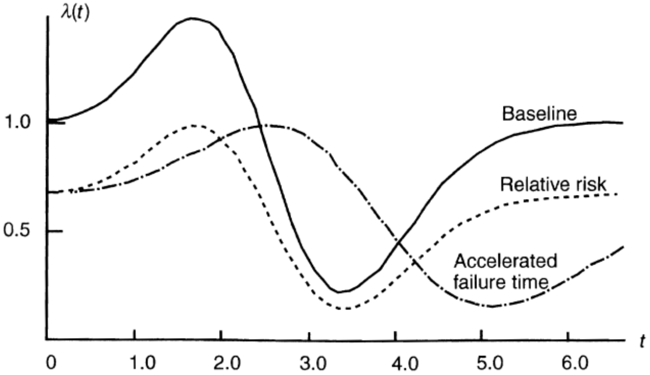


Figure 2.4 The baseline hazard function $\lambda_0(u)$ corresponding to $Z = 0$ is compared to the hazard for $Z = 1$ ($\beta = -\log 1.5$) under a relative risk model and to $z = 1$ ($\beta = \log 1.5$) under an accelerated failure time model.

note

Exponential and Weibull regression models can be considered as special cases of both models.

Discrete failure time models

Discrete failure time?

- Grouping of continuous data due to imprecise measurement;
- Time itself may be discrete
  - e.g., when the response time represents the number of episodes that occur prior to a terminal event.

Discrete regression models?

- Grouped relative risk model;
- Discrete and continuous relative risk model;
- Discrete logistic model.

Discrete regression models

- Grouped relative risk model:

  $$\text{Discrete baseline cumulative hazard function}: \Lambda_0(t) = \sum_{a_i \leq t} \lambda_i,$$

  this model is the uniquely appropriate one for grouped data from the continuous relative risk model.

- Discrete and continuous relative risk model:

  $$d\Lambda(t; x) = \exp(Z^\top \beta) \, d\Lambda_0(t),$$

  which retains the multiplicative hazard relationship.

- Discrete logistic model:

  $$\frac{d\Lambda(t; x)}{1 - d\Lambda(t; x)} = \frac{d\Lambda_0(t)}{1 - d\Lambda_0(t)} \exp(Z^\top \beta),$$

  specifies a linear log odds model for the hazard probability at each potential failure time.


THANK YOU
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